Sharpe's Arithmetic and the Risk Matters Hypothesis

James White, Vladimir Ragulin & Victor Haghani*

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In Lake Wobegon, all the women are strong, all the men are good-looking, and all the children are above average.
- Garrison Keillor

In 1991, William Sharpe made perhaps the strongest argument to date for market capitalization weighted index investing in a three page article titled, "The Arithmetic of Active Investing":

If "active" and "passive" management styles are defined in sensible ways, it must be the case that

(1) before costs, the return on the average actively managed dollar will equal the return on the average passively managed dollar and
(2) after costs, the return on the average actively managed dollar will be less than the return on the average passively managed dollar

These assertions will hold for any time period. Moreover, they depend only on the laws of addition, subtraction, multiplication and division. Nothing else is required.

The key insight of this idea is that, if we add all non-market capitalization weighted portfolios together into one big portfolio, it must be identical to the market capitalization weighted portfolio, i.e. the "market portfolio." While the practical implications of Sharpe's Arithmetic have been debated, its logic has been broadly accepted, and many see it as being one of the main drivers of the massive growth of index investing over the past three decades.[1]

Sharpe's seminal paper landed a body blow on the stock picking industry. Perhaps he felt his argument packed more than enough punch to make investors rethink their stance on stock picking - but whatever his reasons, he stopped short of laying out a corollary to his Arithmetic of Active Management, which is just as powerful an indictment.

The corollary requires a bit more explanation than his main argument, but it is almost as simple and rests on the same basic insight laid out in his 1991 paper, that all active portfolios aggregate to the market portfolio:

(1) the average risk across all actively managed portfolios of stocks will be greater than the risk of the market portfolio, and
(2) the average risk-adjusted excess return across all active portfolios will be less than the risk-adjusted excess return of the market portfolio, before taking account of fees and trading costs

We can see why the average risk across all active portfolios is greater than the risk of the market portfolio by seeing that every active portfolio can be expressed as holding the market portfolio plus an "active exposures" portfolio of longs and shorts in all the constituents, such that the market portfolio plus the active exposures portfolio equals the given active portfolio. Further, each active portfolio requires that there be someone(s) holding an active portfolio with the opposite active exposures, which we'll call the mirror portfolio.

* Victor is the founder and CIO of Elm Partners, and James is Elm's CEO. This not is not an offer or solicitation to invest. **Past returns are not indicative of future performance.** We thank John Campbell, Jeffrey Rosenbluth and Mark Grinblatt for their help and encouragement. All errors are our own.

[1] For discussions of where Sharpe's Arithmetic may not be a good model of reality, see Pedersen (2018), Chen et al. (2006), or Dick-Nielsen (2012) for the analysis of frictions in bond index funds.

The average of the risk of any active portfolio and its mirror will be greater than the risk of the market portfolio. The key to seeing why is to notice that the two portfolios of active exposures have the same risk, but their correlations to the market portfolio will have opposite signs. As a result, when the portfolios are averaged together the correlation terms will cancel each other out, leaving just the extra tracking risk from the active exposures as an addition to the risk of the market portfolio. Since this holds for any active portfolio, it follows that averaging across any number of active portfolios gives the result that the average risk across all active portfolios must be greater than the risk of the market portfolio.

A little algebra shows us that the average of the risk of any arbitrary active portfolio and the risk of the mirror active portfolio must be greater than the risk of the market portfolio:[2]

Risk of the Active Portfolio $= \sigma_m^2 + \sigma_a^2 + 2\rho\sigma_m\sigma_a$

Risk of the Mirror Portfolio $= \sigma_m^2 + \sigma_a^2 - 2\rho\sigma_m\sigma_a$

Risk of the Market Portfolio $= \sigma_m^2$

$$\frac{1}{2}((\sigma_m^2 + \sigma_a^2 + 2\rho\sigma_m\sigma_a) + (\sigma_m^2 + \sigma_a^2 - 2\rho\sigma_m\sigma_a)) > \sigma_m^2$$

$$\sigma_m^2 + \sigma_a^2 > \sigma_m^2$$

$$\sigma_a^2 > 0$$

where σ_m is the standard deviation of returns of the market portfolio, σ_a is the standard deviation of returns of the portfolio of active exposures (i.e. tracking risk), and ρ is the correlation between the returns of the market portfolio and the returns of the portfolio of active exposures.

We cannot easily say how much higher the average risk of active portfolios will be versus the risk of the market portfolio, as it depends on the concentration of each active portfolio. However, we can get a sense for the magnitude by considering randomly constructed portfolios holding different numbers of stocks, such that in aggregate all the portfolios equal the market portfolio.

In the table below, we compare the risk of the market portfolio with the average risk of portfolios randomly constructed of 5, 25, and 100 stocks, selected so that they aggregate as closely as possible to the market portfolio.[3] These concentrated portfolios have between 4% and 30% more risk than the market portfolio (see furthest right column). These active portfolios of N stocks are riskier than one might naively estimate by assuming that portfolio idiosyncratic risk decreases with $\sqrt{\frac{1}{N}}$. This is because 1) many of the idiosyncratic risks of individual stocks are correlated with each other (e.g. through being in the same industry sector or sharing factor exposures), and, 2) the uneven market capitalization weights result in greater concentration in portfolios than would arise from portfolios in which each stock had the same weight.

[2] Please see Appendix for a more rigorous treatment. This result also holds for the average standard deviation of returns, for $-1 < \rho < 1$, but the math is not as neat and tidy as it is for the average variances.

[3] We use the past 10 years of weekly return data, and current weights of the Bloomberg 500 US stock index. Our simulated investor portfolios hold stocks with market-cap weights, which is different from the standard calculation of diversification benefits which assumes equal weights, e.g. Malkiel (2023). Once the stock has been selected, we include it with the weight proportional to its market cap. This is because with the equal-weighted approach, it is not possible for the aggregated holdings to match the market weight of the mega-caps like AAPL by aggregating equal-weighted portfolios of more than 15 stocks, since even if each concentrated portfolio holds AAPL (which it wouldn't), adding them together only gives a 6.7% AAPL weight (= 1/15) for the aggregated portfolio - below the actual 7% weight. With our approach, an investor holds larger positions in the mega-caps, and therefore needs more stocks to achieve the same risk reduction vs. the standard equal-weighted approach.

Concentrated Portfolios v. Aggregated Holdings						
10y History, weekly data, 100k simulated investor portfolios						
# Stocks in a Concentrated Portfolio	Probability of Selecting a Stock	Weight of a Selected Stock	Average Concentrated Portfolio Vol	Aggregated Portfolio Vol	Active - Aggregated Vol	Active vs. Aggregated Relative Risk
5	Equal	Market Cap	22.9	17.6	**5.3**	**30%**
25	Equal	Market Cap	19.7	17.6	**2.1**	**12%**
100	Equal	Market Cap	18.3	17.6	**0.7**	**4%**

If, as was the case in the 1960s, the median number of stocks in an individual's brokerage account was just two, the average riskiness of these highly concentrated portfolios would be 1.5x that of the market portfolio. A more recent 2005 study showed that stock investors with liquid assets over $1mm directly hold on average 15 stocks.[4]

Another viewpoint we can take is to compare the average risk of a set of non-market capitalization weighted ETFs and mutual funds to the risk of the S&P 500 over the past 10 years. Even though these actively-managed funds hold about 200 stocks each, their average risk was 7.4% higher than the risk of the relevant market portfolio. It is interesting to note that Vanguard's growth and value funds - which each own over 200 stocks, and represent mirror active portfolios of each other - have an average risk that is 7% higher than the S&P 500. The table also shows the average risk and return for all the portfolios that are mirrors of the actual ETFs and funds shown in the table. We see that the average risk of these mirror portfolios is 6.2% higher than the risk of the market portfolio, and had a lower compound return than the market portfolio too.

The average return of the actual portfolios and the mirror portfolios is 0.5% lower than the return of the S&P 500. This is the result of two effects: 1) the active portfolios have fees that are 0.26% higher than that of the market portfolio index fund, and 2) the higher average volatility of these active portfolios reduces their compound returns relative to the market portfolio by about 0.25%. (If we included estimated trading costs in the mirror portfolios, their average return would have been even lower.)

Comparing Risk of Select Non-Market Cap Funds, and Their Mirror Portfolios to S&P500 ETF (Nov. 5, 2013 to Nov. 3, 2023 weekly data)

		# stocks	Fees	$BB	Return	Risk
Market Portfolio (S&P500 ETF ticker SPY)			0.09%		11.5%	16.6%
Average for Active Portfolios Below		185	0.35%	64	10.5%	17.8%
% Difference versus Market Portfolio					-8.4%	7.4%
Average for Mirror of Portfolios Below			0.35%		11.4%	17.6%
% Difference versus Market Portfolio					-0.4%	6.2%
VUG	Vanguard Growth Fund	222	0.04%	171	13.4%	19.2%
VTV	Vanguard Value Fund	320	0.04%	140	9.3%	16.1%
VMGAX	Vanguard Mega Cap Growth	88	0.07%	14	14.2%	19.3%
DVY	Select Dividend	99	0.38%	18	8.2%	17.0%
SUSA	Sustainable	182	0.25%	5	11.0%	16.7%
MTUM	Momentum	123	0.15%	8	11.1%	18.6%
SDY	Dividend	121	0.35%	20	8.7%	15.9%
QUAL	Quality	125	0.15%		11.4%	16.8%
RSP	Equal Weight S&P 500	503	0.20%	42	9.4%	18.0%
VWNEX	Vanguard Windsor Admiral	126	0.28%	21	9.3%	19.0%
VWNFX	Vanguard Windsor II Inv	177	0.26%	50	9.5%	17.1%
AGTHX	American Funds Growth Fund	327	0.40%	214	11.1%	18.4%
AWSHX	American Funds Washington	193	0.38%	73	10.2%	15.4%
TRBCX	T. Rowe Price Blue Chip Growth	75	0.71%	98	12.1%	20.5%
PRFDX	T. Rowe Price Equity Income	116	1.23%	16	7.3%	17.3%
FMAGX	Fidelity Magellan Fund	58	0.52%	25	11.3%	18.1%
FCNTX	Fidelity Contrafund	292	0.47%	105	12.5%	17.7%

[4] Stambaugh (2014).

You might say, what's the big deal if the risk on a typical actively managed portfolio is 10% higher than the risk of the market portfolio? Well, we think it is a big deal! Assuming it comprises most of the risky part of a portfolio, to be indifferent between the active portfolio and the market capitalization indexed portfolio, you'd want them to have the same Sharpe ratio. This means the active portfolio would need to have 10% more expected return net of fees in excess of the safe asset than the market portfolio. If, for example, you think the market portfolio offers a 4% return in excess of the safe asset, then the active portfolio would need to offer 0.4% more, or a 4.4% excess return, just to be equally as attractive on a risk-adjusted basis.[5]

For years, investors and commentators have bemoaned the roughly 0.6% per annum difference between the average expense ratios on US actively-managed equity mutual funds and US equity index funds.[6] We think they should be just as concerned, if not more so, by the extra cost of risk involved in holding concentrated portfolios in aggregate. This cost of risk of active management can easily be as large as or, in extreme cases of concentration, dwarf the extra fees that have garnered investor attention for so long.

Conclusion

Vanguard founder John Bogle was profoundly impacted by Sharpe's Arithmetic, which he developed into his "Cost Matters Hypothesis" (CMH) presented in the same journal that published Sharpe's Arithmetic 14 years earlier:[7]

Gross returns in the financial markets minus the costs of financial intermediation equal the net returns actually delivered to investors…To explain the dire odds that investors face in their quest to beat the market, however, we don't need the EMH (Efficient Markets Hypothesis); we need only the CMH [Cost Matters Hypothesis]. *No matter how efficient or inefficient markets may be, the returns earned by investors as a group must fall short of the market returns by precisely the amount of the aggregate costs they incur. It is the central fact of investing.*

In the spirit of the late and great John Bogle, we would like to offer the "Risk Matters Hypothesis" (RMH), as an addition to the EMH and CMH in warning investors of the challenge they face in adding value through stock picking:

The average risk-adjusted excess returns across all active portfolios will be less than the risk-adjusted excess return of the market portfolio, before taking account of fees and trading costs.

As we discuss in more detail in our book, *The Missing Billionaires: A Guide to Better Financial Decisions*, it is natural that investors should and do require compensation for bearing risk. However, all too often we don't adequately account for it in our investment decisions.

If there were no extra fees, taxes or other monetary costs associated with active management, Sharpe's 1991 argument may not have been as influential as it has proved to be. In the past 30 years since Sharpe laid out his arithmetic, there has been a dramatic decrease in the fees charged by active stock managers, commissions for retail stock trades have gone to zero, and the inside bid-ask spread on equities has decreased. Taken together, these have reduced - but not eliminated - the importance of Sharpe's original argument.

However, in the risk corollary to Sharpe's Arithmetic described in this note, active investors are engaged in a negative sum activity even if there are no extra fees involved.[8] Logic dictates that investors cannot in aggregate be rewarded for the extra risk they incur in owning concentrated stock portfolios.

Using Sharpe's insightful observation that the portfolios of all active investors must equal the market portfolio and applying it in the dimension of risk, his original warning still rings true that active stock investors in aggregate need to overcome a substantial threshold of extra return in order to improve their welfare.

[5] You may also ask, what's the big deal about a 10% difference in Sharpe ratio, if you don't expect a higher return on your actively managed portfolio? The answer is that a 10% lower Sharpe ratio causes a 20% reduction in your risk-adjusted return, as we describe in *The Missing Billionaires*, Chapter 5, page 59.

[6] Average Equity and Bond Mutual Fund Expense Ratios Continue to Decline (2022).

[7] Bogle, J. 2015. "The Relentless Rules of Humble Arithmetic." *Financial Analysts Journal*, 61 (6), 22-35.

[8] Ignoring some possible, though hard to observe or heavily weigh, risk transfer arguments.

Appendix: Why the Average Risk of Any Combination of Active Portfolios Which Aggregate to the Market Portfolio is Greater Than the Risk of the Market Portfolio [9]

Suppose we have two investors A and B. A has a portfolio with dollar risk σ_a and B with σ_b. Then the aggregate standard deviation $\sigma_a + \sigma_b$ is greater than that of the combined portfolio whose risk we call σ_c.

$$\begin{aligned}(\sigma_a + \sigma_b)^2 &= \sigma_a^2 + \sigma_b^2 + 2\sigma_a\sigma_b \quad (1)\\ &\geq \sigma_a^2 + \sigma_b^2 + 2\rho\sigma_a\sigma_b \text{ (since } \rho \leq 1) \quad (2)\\ &= \sigma_c^2 \quad (3)\end{aligned}$$

Now suppose that the entire market is comprised of n portfolios each with dollar risk σ_i. Then the aggregate portfolio risk is greater than the combined portfolio risk which in this case is the market risk.

$$\begin{aligned}(\sigma_1 + \sigma_2 + \cdots + \sigma_n)^2 &= \sum_i \sum_j \sigma_i\sigma_j \quad (4)\\ &\geq \sum_i \sum_j \rho_{ij}\sigma_i\sigma_j \text{ (since } \rho_{ij} \leq 1) \quad (5)\\ &= \sigma_m^2 \quad (6)\end{aligned}$$

Further Reading and References

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[9] With thanks to Jeffrey Rosenbluth who provided this concise and elegant proof of the general case.